UNIVERSITIES OF MANCHESTER LIVERPOOL
LEEDS SHEFFIELD AND BIRMINGHAM

JOINT MATRICULATION BOARD

GENERAL CERTIFICATE OF EDUCATION

MATHEMATICS—Paper I

ADVANCED

Friday 4 June 1965 9.30-12.30

Careless and untidy work will be penalized.

Answer **seven** *questions.*

1. (*a*) Solve for *x* and *y* the simultaneous equations

$$axy = 1 \qquad \text{and} \qquad x + \frac{1}{y} = b$$

where *a* and *b* are positive.

if *x* and *y* also satisfy the relation

$$\frac{2}{x} + y = c$$

prove that $(a+1)(2a+1) = abc$

(*b*) If x is real and

$$x^2 + (2-k)x + 1 - 2k = 0$$

show that *k* cannot lie between certain limits, and find these limits.

2. (*a*) Express the following complex numbers in the form a + *ib,* where *a* and *b* are real:

(i) $\dfrac{1-i}{(3-i)^2}$

(ii) $(c+i)^4$, where *c* is real.

(*b*) If: $z = x + iy$ and $z^2 = a + ib$, where *x***,** *y, a, b* are rea1, prove that

$$2x^2 = \sqrt{a^2 + b^2} + a$$

By solving the equation

$$z^4 - 6z^2 + 25 = 0$$

for z^2, or otherwise. express each of the four roots of the equation in the form *x*+ *iy.*

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3. (*a*) Find (i) the sum and (ii) the product of the first *n* terms of the geometric sequence 18, 6, 2,...

(*b*) If $2xy = 1 - x$, show that $|x| < 1$ provided that *y* is either positive or less than -1.

Use the expansions of $\log_e(1+x)$ and $\log_e(1-x)$ to show that

$$\log_e\left(\frac{y+1}{y}\right) = \frac{2}{2y+1} + \frac{2}{3(2y+1)^3} + \frac{2}{5(2y+1)^5} + \ldots$$

for $y > 0$ and for $y < -1$.

From the first three terms of this series, calculate $\log_e 2$ giving your answer to three decimal places.

4. (a) Prove that, in the triangle *ABC.*

$$\frac{b^2 + c^2 - a^2}{\cot A} = \frac{c^2 + a^2 - b^2}{\cot B} = \frac{a^2 + b^2 - c^2}{\cot C}$$

Show that, if cot *A.* cot *B,* cot *C* are in arithmetic progression, then so are a^2, b^2, c^2.

(*b*) Find the solutions between -180^0 and 180^0 of the equation

$$2\cos\theta + 6\sin\theta = 5$$

giving your answers correct to the nearest tenth of a degree.

5. (a) Show that the equation of the tangent at the origin to the circle $x^2 + y^2 = ax + by$ is $ax + by = 0$

The circle $x^2 + y^2 = 15x - 10y = 0$ cuts a circle C at right angles at the origin. If C passes through the point (2, 0), find its equation. -

(b) Show that the equation of the tangent to the parabola $y^2 = 4ax$ at the' point $(at^2 2at)$ is

$$ty = x + at^2$$

Find the equation of the common tangent to the parabolas
$y^2 = 4x$ and $x^2 = 32y$.

6. Show that the equation of the normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point $P(a\cos\phi, b\sin\phi)$ is

$$xa\sin\phi - yb\cos\phi = (a^2 - b^2)\sin\phi\cos\phi$$

If J and K are the feet of the perpendiculars from the centre O to the normal and the tangent at P respectively. find the lengths of OJ and OK. and deduce that the area of the rectangle $OJPK$ is

$$\left|\frac{ab(a^2 - b^2)}{a^2\tan\phi + b^2\cot\phi}\right|$$

If the normal meets Ox at G and Oy at H. show that the locus of the mid-point of GH as φ varies is an ellipse.

7. (*a*) Find the values of *x* for which the function

$$y = x^3 + 3x^2 - 9x$$

has its maximum and minimum values.

Find the coordinates of the maximum and minimum points, and of the point of inflexion on the graph of the function. Sketch the graph.

(*b*) Find the volume generated by the rotation through one revolution about the x-axis of the region between the x-axjs and that part of the curve $y = 2\cos x - 1$ for which $|x| < \pi$ and $y > 0$.

8. (a) A tower stands on level ground. A man observes that the top of the tower is at an elevation α to the horizontal from a fixed point on the ground. He walks a distance *x* straight towards the tower, and finds that the elevation is now θ. Express the height *h* of the tower in terms of α, *x* and θ.

Find $dx/d\theta$ in terms of *h* and θ and show that, if θ increases by ε as *x* increases by *z*. where ε and *z* are small, then

$$h \approx \frac{z}{\varepsilon} \sin^2 \theta$$

(*b*) A curve is described by the equation

$$y^3 = e^x y + e^{2x}$$

Obtain an expression for *dy/dx.*

If (*a*, *b*) is the point of the curve where *dy/dx* = 0, prove that $b = -2e^a$ and hence find *a* and *b.*

9. (a) Prove that

$$\int_0^1 \frac{4x+5}{(x+1)(2-x)}\,dx = \frac{14}{3}\log_e 2$$

(b) Evaluate

$$\int_0^{\pi/2} x\cos x\; dx$$

(c) By means of the substitution $u = 1 + x^2$, or otherwise, evaluate

$$\int_0^1 \frac{x^3 dx}{(1+x^2)^{3/2}}$$